SELF-SIMILAR PROBLEM OF THE DECAY OF A TWO-DIMENSIONAL DISCONTINUITY

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The plane problem of the decay of an arbitrary two-dimensional discontinuity for the gasdynamics equations is considered. The initial surface of the discontinuity is assumed to have the shape of an angle close to π. The existence and uniqueness of the solutions of the problem in a linear formulation are proved.

Linear problems on the diffraction and reflection of shocks have been considered in [1-3]. The problem of the decay of a two-dimensional discontinuity reduces to a new boundary-value problem for equations of mixed type with discontinuous coefficients.

1. FORMULATION OF THE PROBLEM

Let some curve Γ separate a plane into two parts, D_0, D_1. Two polytropic gases in states characterized by the constant parameters

$$\begin{aligned} &u = u_1, \quad v = v_1, \quad p = p_1, \quad \rho = \rho_1, \quad S = S_1, \quad \gamma = \gamma_1, \quad (x, y) \in D_1 \\ &u = u_0, \quad v = v_0, \quad p = p_0, \quad \rho = \rho_0, \quad S = S_0, \quad \gamma = \gamma_0, \quad (x, y) \in D_0 \end{aligned} \tag{1.1}$$

are in D_0 and D_1 at the time $t = 0$.

The baffle Γ vanishes at the time $t = 0$. It is required to describe the gas motion.

The solution of this problem is known in the particular case when the initial surface of discontinuity is a straight line. Here the solution of the problem of the decay of the discontinuity can be constructed in the class of self-similar solutions of the one-dimensional gasdynamics equations. It is evidently impossible to construct the solution of the problem in the class of self-similar solutions for an arbitrary curve Γ.

The necessary condition for self-similarity is invariance of the initial data of the problem relative to the transformation of the independent variables x,y corresponding to the infinitesimal operator [4]

$$x\partial / \partial x + y\partial / \partial y$$

This condition is satisfied if and only if the initial surface of discontinuity Γ has the shape of an angle. In this case the problem of seeking the self-similar solution describing the two-dimensional decay of a discontinuity originates.

The decay of a discontinuity symmetric with respect to the bisectrix Γ_1 of the angle Γ is later examined. By virtue of symmetry on Γ_1 the condition of impenetrability is satisfied. Let us introduce a fixed x,y coordinate system in the flow plane so that at the time $t = 0$ the origin would coincide with the vertex of the angle and the y axis would be directed along the side Γ. Let Γ_1 be given by the equation $y = -x \operatorname{tg} \alpha$ in this coordinate system. Then the initial data (1.1) should satisfy the relationships $v_1 = -u_1 \operatorname{tg} \alpha$, $v_0 = -u_0 \operatorname{tg} \alpha$. Without limiting the generality, we can consider that $u_0 = 0$, $p_1 \geq p_0$.

If new dependent and independent variables corresponding to the conical flows

$$\xi = x / t, \quad \eta = y / t, \quad \mathbf{U} = (U, V) = (u - \xi, v - \eta)$$

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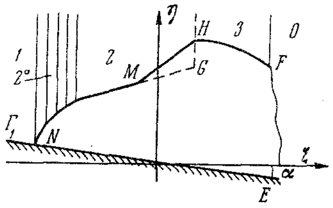


Fig. 1

are introduced, then the system of gasdynamics equations will be reduced to

$$(\mathbf{U}\cdot\nabla)\,\mathbf{U}+\rho^{-1}\nabla p+\mathbf{U}=0$$
$$(\mathbf{U}\cdot\nabla\rho)+\rho\,(\operatorname{div}\mathbf{U}+2)=0,\quad (\mathbf{U}\cdot\nabla S)=0 \tag{1.2}$$

This system is hyperbolic for $|\mathrm{U}|^2 > C^2 = \partial p/\partial\rho$ and elliptic for $|\mathrm{U}|^2 < C^2$.

For large η the decay of the discontinuity is described by the known one-dimensional solution. Different configurations of the one-dimensional decay of a discontinuity are hence possible depending on the magnitudes of the constants prescribing the initial state (1.1). Following the terminology in the book [5], let us designate that decay of a one-dimensional discontinuity such that the shock goes into D_0 and the rarefaction wave into D_1 as the configuration A. The configuration B corresponds to two shocks going into D_0 and D_1, and the configuration C to two rarefaction waves.

The appropriate inequalities for the initial data (1.1) which will assure realization of the configurations mentioned are presented in [5]. Knowing the one-dimensional solutions, we can construct the boundary of the domain in which the flow will be essentially two-dimensional.

For large η let a one-dimensional decay of the discontinuity corresponding to configuration A occur (Fig. 1). The solution is constructed from the simple Riemann wave 2° and the two constant flows adjoining the contact discontinuity. From the intersection of Γ_1 with the forward front of the simple wave let us draw a characteristic of the system (1.2) in the known solution 2°, 2 to intersect the front of the contact discontinuity at the point G. In the constant solution 3 let us construct a circle $U^2 + V^2 = C^2$ intersecting the front of the contact discontinuity at the point H. If $\eta_H > \eta_G$ (as holds in Fig. 1), then a characteristic is drawn from the point H along the constant solution 2 to the intersection with the characteristic NG. The known boundary NMHF and the unknown shock front FE close the flow domain of a double-wave type. The boundary of the mentioned domain can be constructed analogously in the remaining cases also. A definite boundary-value problem for the system (1.2) with the unknown boundaries, the shock and contact discontinuity fronts, originates in the domain NMHFE.

Let us consider the problem of the decay of a discontinuity in a linear formulation by assuming the angle α small. Let us take the one-dimensional decay of a discontinuity for $\alpha = 0$ as the fundamental solution on which the linearization is carried out.

Configuration A. Some possible forms of the perturbed flow domain for the linear problem are represented in Figs. 2a, b, and c. The flow can be considered potential in the domain LMN. Let us introduce the flow potential by means of the formulas $\varphi_\xi = \mathrm{U}$, $\varphi_\eta = \mathrm{V}$ and let us represent the function φ as

$$\varphi = \varphi_0 + \alpha\psi$$

where φ_0 is the potential corresponding to the fundamental solution. An equation for the perturbation potential ψ is obtained after linearization:

$$\frac{1}{2}\left[\eta^2-\left(\frac{2C_1}{\gamma_1+1}-\frac{\gamma_1-1}{\gamma_1+1}(\xi-u_1)\right)^2\right]\psi_{\eta\eta}-\left[\frac{2C_1}{\gamma_1+1}-\frac{\gamma_1-1}{\gamma_1+1}(\xi-u_1)\right]\times$$
$$\times\,\eta\psi_{\xi\eta}-\frac{\gamma_1-1}{\gamma_1+1}\eta\psi_\eta+\left[\frac{2C_1}{\gamma_1+1}-\frac{\gamma_1-1}{\gamma_1+1}(\xi-u_1)\right]\psi_\xi+\frac{\gamma_1-1}{\gamma_1+1}\psi=0 \tag{1.3}$$

The boundary conditions for the function ψ are the following: $\psi = -u_1\eta$ on the characteristic LM, and $\psi_\eta = -2(\gamma_1+1)^{-1}(C_1+\xi) - (\gamma_1-1)(\gamma_1+1)^{-1}u_1$ for $\eta = 0$. Linearizing the system (1.2) in the constant solutions 2, 3 results in the equations

$$(\mathbf{x}_j\cdot\nabla)\,\mathbf{u}^j=\nabla p^j,\quad (\mathbf{x}_j\cdot\nabla)\,\rho^j=\operatorname{div}\mathbf{u}^j,\quad (\mathbf{x}_j\cdot\nabla)\,p^j=\operatorname{div}\mathbf{u}^j \tag{1.4}$$

Here U^j, p^j, ρ^j $(j = 2, 3)$ are the desired dimensionless perturbations defined as follows:

$$\mathbf{u}=\mathbf{u}_j+\alpha C_j\mathbf{u}^j,\quad p=p_j+\alpha\rho_jC_j^2p^j$$
$$\rho=\rho_j(1+\alpha\rho^j),\quad x_j=(\xi-U_0)/C_j,\quad y_j=\eta/C_j$$

where p_j, ρ_j, C_j, $\mathrm{U}_j = (U_0, 0)$ are the gas parameters in the fundamental constant solutions. An equation for the function p^j follows from the system (1.4):

$$(x_j^2-1)\,p^j_{x_jx_j}+2x_jy_jp^j_{x_jy_j}+(y_j^2-1)\,p^j_{y_jy_j}+2x_jp^j_{x_j}+2y_jp^j_{y_j}=0 \tag{1.5}$$

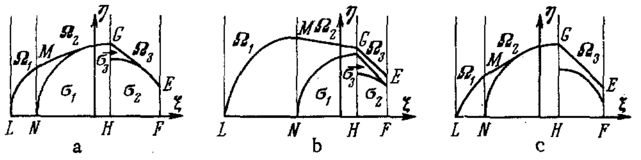


Fig. 2

The boundary conditions for the system (1.4) are

$$p^2 = u^2 = \rho^2 = 0, \quad v^2 = -u_1 / C_2, \quad (\xi, \eta) \in MG$$
$$v^2 = -U_0 / C_2, \; p_{y_2}{}^2 = 0, \; y_2 = 0, \; p^3 = u^3 = \rho^3 = v^3 = 0$$
$$(\xi, \eta) \in GE, \; v^3 = -U_0 / C_3, \; p_{y_3}{}^3 = 0, \; y_3 = 0$$

Let the front of the contact discontinuity in the perturbed domain be given by the equation

$$\xi = U_0 + \alpha f(\eta)$$

The conditions on the contact discontinuity yield the following boundary conditions:

$$\gamma_1 p^2 = \gamma_0 p_3, \; C_2 u^2 = C_3 u^3 = f(\eta) - \eta f'(\eta), \; x_2 = x_3 = 0$$

Using (1.4), the last relation can be written as

$$C_2 p_{x_2}{}^2 = C_3 p_{x_3}{}^3 \quad (y_2 = C_2^{-1} C_3 y_3)$$

Let the perturbed portion of the shock front be given by the equation

$$x_3 = k_3 + \alpha \psi_3(y_3) \left(k_3 = \frac{D_3 - U_0}{C_3} = \left[\frac{(\gamma_0 - 1) M_3^2 + 2}{2\gamma_0 M_3^2 - \gamma_0 + 1}\right]^{1/2}, \; M_3 = \frac{D_3}{C_0}\right)$$

where D_3 is the velocity of the unperturbed shock front. Using the Hugoniot relationship on the shock, the u^3, v^3, p^3, ρ^3 can be calculated on the boundary EF:

$$u^3 = \frac{2}{\gamma_0 + 1} \frac{M_3^2 + 1}{M_3^2} (\psi_3(y_3) - y_3 \psi'(y_3)), \quad v^3 = -\frac{2k_3 (M_3^2 - 1)}{(\gamma_0 - 1) M_3^2 + 2} \psi_3'(y_3)$$
$$p^3 = \frac{4k_3}{\gamma_0 + 1} (\psi_3(y_3) - y_3 \psi'(y_3)), \quad \rho^3 = \frac{4}{(\gamma_0 + 1) k_3 M_3^2} (\psi_3(y_3) - y_3 \psi'(y_3)) \tag{1.6}$$

From (1.4), (1.6) we obtain

$$y_j (k_j^2 - 1) p_{x_j}{}^j + [(L_j + k_j) y_j^2 - N_j k_j] p_{y_j}{}^j = 0$$
$$x_j = k_j \; (j = 3), \; L_3 = 2k_3^{-1} M_3^{-2} (M_3^2 + 1),$$
$$N_3 = (\gamma_0 + 1)(M_3^2 - 1)[2(\gamma_0 - 1) M_3^2 + 4]^{-1} \tag{1.7}$$

Still another smooth-shock-front condition at the point E should be satisfied on EF:

$$\int_{EF} p_{y_3}{}^3 (k_3, t) \frac{dt}{t} = \frac{4k_3}{\gamma_0 + 1} \tag{1.8}$$

Configuration B (Figs. 3a, b, c). In this case the linearized equations are in the form (1.4), (1.5) for j = 2, 3. The boundary conditions in the domain Ω_3 are the same as in the case of configuration A. The conditions on the boundaries MG, GH, NH in the domain Ω_2 also do not change. The conditions on the boundary MH

$$x_2 = k_2 = -[(\gamma_1 - 1) M_2^2 + 2]^{1/2} [2\gamma_1 M_2^2 - \gamma_1 + 1]^{-1/2}, \; M_2 = |D_2 - u_1| / C_1$$

have the same form as the conditions on the boundary $x_3 = k_3$ if γ_0 is replaced everywhere in (1.6)-(1.8) by γ_1 and the subscript 3 by 2, with the exception of the second condition (1.6), which is in this case

$$v^2 = -\frac{2k_2 (M_2^2 - 1)}{(\gamma_1 - 1) M_2^2 + 2} \psi_2'(y_2) - \frac{u_1}{C_2}$$

Configuration C (Figs. 4a, b). The boundary conditions and equations remain the same in the domains Ω_1, Ω_2 as in the case of configuration A. The equation for the perturbation potential and the boundary conditions in the domain Ω_4 agree with the equation and conditions which are valid in Ω_1 if $u_1 = 0$ is substituted, and ξ, C_1, γ_1 are replaced by $-\xi$, C_0, γ_0. The same boundary conditions are valid on the remaining boundaries GE, GH, and HQ as in the case of configuration B.

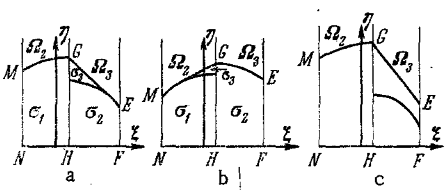


Fig. 3

2. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE LINEAR PROBLEM

The existence and uniqueness of the solution will be proved under the following constraints on the parameters of the problem:

$$C_2C_3^{-1} < (1-k_3^2)^{-1/2}$$

in the case of configuration A and

$$(1-k_2^2)^{1/2} < C_2C_3^{-1} < (1-k_3^2)^{-1/2}$$

in the case of configuration B.

For such a choice of the initial data (1.1) the characteristic issuing from the intersection of the line of degeneration $x_j^2 + y_j^2 = 1$ of one of the equations (1.5) with the line GH terminates on the line of degeneration of the second equation (Figs. 2, 3a, b). Otherwise, this characteristic arrives on the shock (Figs. 2, 3c).

The solution of the linearized problem outside the domains σ_1, σ_2, σ_3 is found uniquely in explicit form. As has been shown in [6], equation (1.3) is integrated in quadratures, and the mixed problem in the domains Ω_1 and Ω_4 reduces to the solution of an Abel integral equation of the first kind, solvable explicitly. By the change of variables

$$z = \theta + \arccos r_j^{-1}, \quad \tau = \theta - \arccos r_j^{-1}$$
$$(\theta = \operatorname{arc\,tg}(y_j / x_j), \quad r_j^2 = x_j^2 + y_j^2) \tag{2.1}$$

(1.5) is reduced in the hyperbolic domain to the wave equation $p_{Z\tau}{}^j = 0$. Knowing the general form of the solution of (1.5), the solution of the linear problem outside the domains σ_1, σ_2, σ_3 can be constructed explicitly.

Therefore, the proof of the existence and uniqueness of the solution of the problem of the decay of a discontinuity reduces to the proof of these facts for the auxiliary problem consisting of the following: find a pair of functions $P = (p^2(x_2, y_2), p_y(x_3, y_3))$ defined for $x_2 \le 0$, $x_3 \ge 0$, respectively, which will satisfy (1.5) (j = 2, 3), be continuous in the closed domains σ_1, σ_2, σ_3, twice continuously differentiable in the domains σ_2, σ_3, and continuously differentiable in the domain σ_3 such that the derivatives $p_{x_j}{}^j$, $p_{y_j}{}^j$ will exist and be continuous for $x_j = 0$ and, in the case of configurations A and B, for $x_j = k_j$. The functions $p^j(x_j, y_j)$ should satisfy the following boundary conditions for A, B, C:

$$p_{y_j}{}^j = 0, \; y_j = 0, \; p^j|_l = \chi_j(y_j)$$
$$\gamma_1 p^2 = \gamma_0 p^3, \; C_2 p_{x_2}{}^2 = C_3 p_{x_3}{}^3, \; x_2 = x_3 = 0 \tag{2.2}$$

The following conditions are posed for A, B: condition (1.7) (j = 3) and the integral condition

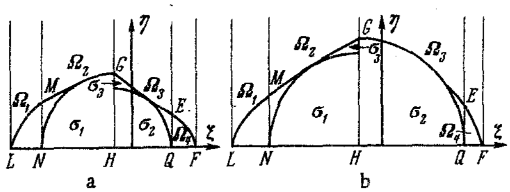
$$\delta_1(P) = \int_0^{k_3'} p_{y_3}{}^3(k_3, t)\frac{dt}{t} = T_1 \tag{2.3}$$

should be satisfied on the boundary $x_3 = k_3$.

In the case of configuration B, condition (1.7) (j = 2) is also satisfied on the boundary $x_2 = k_2$; moreover

$$\delta_2(P) = \int_0^{k'_2} p_{y_2}{}^2(k_2, t)\frac{dt}{t} = T_2 \tag{2.4}$$

Here l is part of the domain boundary $\sigma_1 \cup \sigma_2 \cup \sigma_3$ consisting of segments of the line of degeneration of the type of (1.5) and of the characteristic bounding the domain σ_3; T_1 and T_2 are given constants, $\chi_j(y_j)$ are functions differentiable along l with Hölder continuous derivative, and $k_j' = \sqrt{1-k_j^2}$.



a b

Fig. 4

Theorem 1. The solution of the problem (1.7), (2.2)-(2.4) is unique under the assumptions made above.

By virtue of the linearity of the problem it is sufficient to show that the homogeneous problem has just a trivial solution. Let $\chi_j(y_j) = T_1 = T_2 = 0$. Let us continue the solution in a domain symmetric relative to the axes $y_2 = y_3 = 0$ by assuming $p^j(x_j, y_j) = p^j(x_j, -y_j)$. Let

$$p_* = \begin{cases} \gamma_0 p^3(x_3, y_3), & x_3 \geqslant 0 \\ \gamma_1 p^2(x_2, y_2), & x_2 \leqslant 0 \end{cases}$$

The function p_* cannot reach a global positive maximum or negative minimum within the domain of ellipticity σ_1, σ_2 since otherwise p_* = const in any internal subdomain, and therefore $p_* = 0$. The extremum is not reached on the line $x_2 = x_3 = 0$ either. Indeed, in the cases corresponding to Figs. 2, 3, and 4a, at the point of the positive maximum, $p_{x_3}^3 \leq 0$ and $p_{x_2}^3 > 0$ according to the Hopf maximum principle [7, 8], which contradicts condition (2.2). Now p_* = const along the first family characteristic intersecting the boundaries $x_2 = x_3 = 0$ in the domain σ_3. Therefore, the extremum cannot be reached in the closed domain σ_3 either. In the case of the configuration C the uniqueness of the solution is proved since the function $p_* = 0$ on the remaining boundaries. In the case of configurations A and B the function p_* cannot reach the extremum even at the points $x_3 = k_3$, $y_3 \neq 0$ since it follows from condition (1.7) and the equality $p_{y_3}^3 = 0$ that $p_{x_3}^3 = 0$ while $p_{x_3}^3 \neq 0$ at the extremum points according to the Hopf maximum principle. Let us assume that the extremum is reached at the point $x_3 = k_3$, $y_3 = 0$. Let us put $p_0^3 = p^3(k_3, 0)$. Integrating (2.3) by parts, we obtain

$$-\frac{p_0^3}{k_3'} + \int_0^{k_3'} \frac{p^3 - p_0^3}{t^2}\, dt = 0 \tag{2.5}$$

But both terms in (2.5) have the same sign in the case of a positive maximum or negative minimum, from which we obtain that $p_0^3 = 0$. In the case of configuration B it is shown analogously that the function p_* cannot reach the extremum for $x_2 = k_2$. The theorem is proved.

Corollary 1. For any two solutions P_1 and P_2 of the homogeneous problem (1.7), (2.2) the linear dependence of the solutions P_1 and P_2 follows from the linear dependence of the vectors $\delta(P_1) = (\delta_1(P_1), \delta_2(P_1))$, $\delta(P_2) = (\delta_1(P_2), \delta_2(P_2))$.

The existence of the solution is proved by reducing the problem to some singular integral equation. Let us give on the "merger" line $x_2 = x_3 = 0$

$$p_{y_2}{}^2 = \omega(y_2), \quad p_{y_3}{}^3 = \gamma_1 \gamma_0^{-1} C_3 C_2^{-1} \omega(C_3 C_2^{-1} y_3) \tag{2.6}$$

Here $\omega(y)$ is an arbitrary function ($\omega(0) = 0$). One of the conditions (2.2) will hence be satisfied. The mixed problem (2.2), (2.6) in the domain σ_3 can be solved explicitly; in particular, the quantities $p_{x_j}^j$ can be calculated for $x_j = 0$ and p_{θ_j} for $r_j = 1$.

By the change of variables

$$\theta = \theta', \quad r_j = \frac{2R_j}{R_j^2 + 1} \tag{2.7}$$

(1.5) is reduced to the Laplace equation in the domain of ellipticity. It follows from (2.1), (2.7) that the derivatives $p_{r_j}^j$ become infinite on the order of $|1 - r_j|^{-1/2}$ on the lines of degeneration.

Let us examine the analytic function

$$\Phi^j(\zeta_j) = p_{\xi_j}{}^j - i p_{\eta_j}{}^j \quad (j = 2, 3)$$

$$\xi_j = R_j \cos\theta, \quad \eta_j = R_j \sin\theta, \quad \zeta_j = \xi_j + i\eta_j$$

By virtue of the boundary conditions Hilbert problems originate for Φ^2 and Φ^3 in the domains σ_1 and σ_2. The solution of the Hilbert problem is sought in the class of solutions bounded at points of discontinuity of the coefficients of the boundary condition. The solution in a domain of quadrant type (in the domain σ_1 in Fig. 2a) is hence determined uniquely in explicit form, and in a domain of quadrangle type (the domain σ_1 in Fig. 2a) to the accuracy of an arbitrary constant. Evaluating $p_{x_j}^j$ for $x_j = 0$ by means of the solution found and satisfying the second condition in (2.2) on the line $x_2 = x_3 = 0$, we obtain a singular integral equation to determine the function $\omega(y)$.

Omitting the intermediate calculations, let us write down the integral equation for the case corresponding to Fig. 2a:

$$
\begin{aligned}
&Ts^{-2}\sqrt{s^2y^2-1}\left[\theta\left(y-\frac{1}{s}\right)-\theta(y-1)\right]\omega(y)+\frac{1}{\pi}\Big[\sqrt{1-y^2}+\\
&+Ts^{-2}\sqrt{1-s^2y^2}\left[\theta(y)-\theta\left(y-\frac{1}{s}\right)\right]\Big]\int_0^1\frac{2\tau\omega(\tau)}{\tau^2-y^2}\,d\tau+\frac{Ts^{-2}}{\pi}\sqrt{1-s^2y^2}\times\\
&\times\left[\theta(y)-\theta\left(y-\frac{1}{s}\right)\right]\int_0^1\left[\frac{Y(sy)}{Y(st)}\frac{2sg(st)g'(st)}{g^2(st)-g^2(sy)}-\frac{2t}{t^2-y^2}\right]\omega(t)\,dt=\\
&=\frac{1}{\pi}\sqrt{1-y^2}\int_0^1\frac{2t\chi_2'(t)}{1-t^2y^2}\,dt+\frac{1}{s\pi}\sqrt{1-s^2y^2}\left[\theta(y)-\theta\left(y-\frac{1}{s}\right)\right]\int_{s^{-1}}^1\times\\
&\times\frac{\sqrt{s^2-1}}{\sqrt{s^2t^2-1}}\frac{2sg(st)g'(st)}{g^2(st)-g^2(sy)}\frac{Y(sy)}{Y(st)}\left[\frac{1}{\tau_+(t,s)}\chi_3'\left(\sqrt{\frac{\tau_-(t,s)}{\tau_+(t,s)}}\right)+\right.\\
&\left.+\frac{1}{\tau_-(t,s)}\chi_3'\left(\sqrt{\frac{\tau_+(t,s)}{\tau_-(t,s)}}\right)\right]dt+\frac{2\sqrt{s^2-1}}{s\tau_-(y,s)}\left[\theta\left(y-\frac{1}{s}\right)-[\theta(y-1)\right]\chi_3'\times\\
&\times\left(\sqrt{\frac{\tau_+(x,s)}{\tau_-(y,s)}}\right)-Ds^{-1}\sqrt{1-s^2y^2}\,[\theta(y)-\theta(y-s^{-1})]\,Y(sy)
\end{aligned}
\tag{2.8}
$$

Here

$$
s=C_2C_3^{-1},\quad T=\gamma_0\gamma_1^{-1},\quad q=(1-k_3)(1+k_3)^{-1}
$$

$$
d=(\varepsilon-1)(\varepsilon+1)^{-1},\quad b=(\beta-1)(\beta+1)^{-1}
$$

D is an arbitrary constant, $\theta(y)$ is the Heaviside function, $\vartheta_2(z,q)$, $\vartheta_3(z,q)$ are elliptic theta-functions, and the constants ε and β satisfy the system of equations

$$
\frac{1}{\varepsilon+\beta}=L_3-N_3\frac{k_3}{1-k_3^2},\qquad \frac{\varepsilon\beta}{\varepsilon+\beta}=\frac{N_3k_3}{1-k_3^2}
$$

Moreover

$$
\tau_{\pm}(t,s)=s^2t+1\pm\sqrt{s^2t^2-1}\,\sqrt{s^2-1}
$$

$$
g(t)=-2g^{1/4}\frac{\vartheta_2(0,q)}{\vartheta_3(0,q)}\,t\prod_{n=1}^{\infty}\frac{(1-q^{2^n})^2+4q^{2^n}t^2}{(1-q^{2^{n-1}})^2+4q^{2^{n-1}}t^2}
$$

$$
Y(t)=2q^{1/4}\frac{\vartheta_2^2(0,q)}{\vartheta_2^2(0,q)}\prod_{n=1}^{\infty}\frac{[(1+q^{2n-1})^2-4q^{2^{n-1}}t^2]\,[(1+q^{2^n})^2-4q^{2^n}t^2]}{[(1+dq^{2^{n-1}})^2-4dq^{2^{n-1}}t^2]\,[(1+bq^{2^{n-1}})^2-4bq^{2^{n-1}}t^2]}
$$

In the general case the singular integral equation is

$$
L\omega(y)=G(y)+D_1H_1(y)+D_2H_2(y) \tag{2.9}
$$

where L is a singular operator with coefficients continuous in the segment [0, 1], G(y) is a linear operator over the known boundary data, $H_1(y)$ and $H_2(y)$ are Hölder continuous functions independent of the boundary data, and D_1 and D_2 are arbitrary constants. In the case of configuration A, $H_1(y)\not\equiv 0$, $H_2(y)\equiv 0$, in the case of configuration B, $H_1(y)\not\equiv 0$, $H_2(y)\not\equiv 0$, and in the case of configuration C, $H_1(y)\equiv 0$, $H_2(y)\equiv 0$. The index of the integral equation (2.9) in the class of solutions bounded at the point $y = 0$ is zero.

The proof of the existence of the solution of the original problem reduces to proving the existence of the solution of the integral equation (2.9) and to determining constants D_1 and D_2 such that the solution of the corresponding Hilbert problem would satisfy the conditions (2.3), (2.4). The arbitrary constants D_1 and D_2 originate in the solution of the Hilbert problem as was mentioned above. Let $P = \Psi(\omega)$ be the operator setting the solution of (2.9) for fixed D_1 and D_2 in correspondence with the solution of the Hilbert problem with the same constants D_1 and D_2.

Theorem 2. The solution of the problem (1.7), (2.2)-(2.4) exists under the assumptions made above.

Configuration C. The homogeneous integral equation (2.9) corresponds to the homogeneous problem (2.2). By virtue of Theorem 1 and the equality $\varkappa = 0$, (2.9) is solvable for any right-hand side. In this case the existence of the solution is proved.

Configuration A. Let the homogeneous equation (2.9) have only a trivial solution. Then the inhomogeneous equation is solvable for any right-hand side. Equation (2.9) with the right-hand side $D_1H_1(y)$ corresponds to the homogeneous problem (1.7), (2.2). Let $\omega_1(y) = L^{-1}H_1(y)$, $P = \Psi(\omega_1)$. Here L^{-1} is a linear operator setting its solution in correspondence with the right-hand side of (2.9). It follows from Theorem 1 that

$$\delta_1(P_1) \neq 0 \tag{2.10}$$

Let us put

$$\omega(y) = D_1\omega_1(y) + L^{-1}G(y), \quad P = \Psi(\omega) = D_1P_1 + P_2$$

By satisfying condition (2.3) we obtain a linear equation in the constant D_1:

$$\delta_1(P) = D_1\delta_1(P_1) + \delta_1(P_2) = T_1$$

which is solvable uniquely because of (2.10).

Let $\omega_0(y)$ be a nontrivial solution of the homogeneous integral equation. According to Corollary 1, any other solution of this equation depends linearly on $\omega_0(y)$. Then the dimensionality of the space of solutions of the adjoint homogeneous equation in the adjoint class of solutions [9] also equals one. Let $\omega_*(y)$ be a nontrivial solution of the adjoint homogeneous equation. Compliance with the solvability condition

$$(\omega_*, F) = \int_0^1 F(t)\,\omega_*(t)\,dt = 0 \tag{2.11}$$

is necessary and sufficient for the solvability of (2.9) with the right side $F(y)$.

It is asserted that $(\omega_*, H_1) \neq 0$. If this is not so, then (2.9) with the right side $H_1(y)$ is solvable. But by virtue of Corollary 1 the functions $\omega(y) = L^{-1}H_1(y)$ and $\omega_0(y)$ are linearly dependent, which is impossible. The solvability condition for the integral equation can be satisfied by selecting the constant D_1. The general solution of (2.9) is

$$\omega(y) = L^{-1}(D_1H_1(y) + G(y)) + D_0\omega_0(y)$$

where D_0 is an arbitrary constant. By virtue of (2.9) the condition (2.3) can be satisfied by selecting D_0.

Configuration B. Let the homogeneous equation (2.9) have just a trivial solution. Let us put

$$\omega_1(y) = L^{-1}H_1(y), \ \omega_2(y) = L^{-1}H_2(y), \ P_1 = \Psi(\omega_1), \ P_2 = \Psi(\omega_2)$$

We obtain from the linear independence of the functions H_1 and H_2 and Corollary 1

$$\det(\delta_k(P_l)) \neq 0 \quad (k = 1, 2;\ l = 1, 2) \tag{2.12}$$

The general solution of (2.9) is

$$\omega(y) = D_1\omega_1(y) + D_2\omega_2(y) + L^{-1}G(y)$$

Let $P = \Psi(\omega)$. By satisfying conditions (2.3), (2.4) we obtain a system of two linear equations solvable by virtue of (2.12). In this case, the solution exists.

Let the dimensionality of the space of solutions of the homogeneous equation be one, and $\omega_*(y)$ the solution of the adjoint homogeneous equation as in the previous case. It can be shown that either $(H_1, \omega_*) \neq 0$ or $(H_2, \omega_*) \neq 0$. For definiteness let

$$(H_1, \omega_*) = a_1 \neq 0, \quad (H_2, \omega_*) = b_1$$

Let us put

$$H_3(y) = b_1H_1(y) + a_1H_2(y), \ H_4(y) = b_1H_1(y) - a_1H_2(y) \ \text{for} \quad b_1 \neq 0;$$
$$H_3(y) = H_1(y), \ H_4(y) = H_2(y) \ \text{for} \ b_1 = 0$$

Equation (2.9) can be written in the equivalent form

$$L\omega(y) = D_3H_3(y) + D_4H_4(y) + G(y) \tag{2.13}$$

The condition for solvability of (2.13)

$$D_3(H_3, \omega_*) + (G, \omega_*) = 0$$

can be satisfied by selecting D_3. The general solution of (2.13) is

$$\omega(y) = L^{-1}(D_3H_3(y) + G(y)) + D_4L^{-1}H_4(y) + D_5\omega_0(y)$$

where D_4 and D_5 are arbitrary constants.

Let

$$P_0 = \Psi(\omega_0), \quad P_1 = \Psi(L^{-1}H_4)$$

According to Corollary 1

$$\det(\delta_k(P_l)) \neq 0 \quad (k = 1, 2;\ l = 0, 1)$$

Therefore, conditions (2.3), (2.4) can even be satisfied in this case.

It is shown by analogous reasoning that the solution of the problem also exists in the case when the dimensionality of the space of solutions of the adjoint homogeneous equation is two. It cannot be greater than two because of Corollary 1. The theorem is proved.

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LITERATURE CITED

1. M. J. Lighthill, "Diffraction of blast," Proc. Roy. Soc., Ser. A, 200, 554-565 (1950).
2. J. L. Smyrl, "The impact of a shock-wave on a thin two-dimensional airfoil moving at supersonic speed," J. Fluid Mech., 15, Pt. 2 (1963).
3. S. M. Ter-Minassiants, "The diffraction accompanying the regular reflection of a plane obliquely impinging shock from the walls of an obtuse wedge," J. Fluid Mech., 35, Pt. 2 (1969).
4. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Izd. SO AN SSSR, Novosibirsk (1962).
5. B. L. Rozhdestvenskii and N. N. Yanenko, Systems of Quasilinear Equations and Their Applications to Gasdynamics [in Russian], Nauka, Moscow (1968).
6. V. M. Teshukov, "On the problem of an angular piston," Zh. Prikl. Mekhan. i Tekhn. Fiz., No. 3 (1969).
7. R. Courant, Partial Differential Equations [Russian translation], Mir, Moscow (1964).
8. E. T. Whittaker and G. N. Watson, Modern Analysis, Pt. 2, Cambridge Univ. Press (1927).
9. N. I. Muskhelishvili, Singular Integral Equations [in Russian], Nauka, Moscow (1968).